Converging Solution of Boundary Value Time Fractional Heat Equation

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Abstract:

In this paper we will study the converging solution of boundary value time fractional heat equation involving fractional derivative of order lies between (0,2) and analyze the temperature effect on the surface. The solution which is the result of different transform like Laplace, Hankel and Fourier at a boundary we will find the equality for parabolic and derivative limit. The temperature level for the inner and outer surfaces and upper and lower surfaces will keep different while obtaining the solution of boundary value time fractional heat equation.

Keywords:

Heat equation, nonlinear boundary value problem, heat conduction, Mittag-Leffler functions, converging solution.

Introduction

The concept of generalization of classical coupled thermos elasticity, thermos elastic behavior of the material without energy dissipation with linear and nonlinear theories, thermal stresses of heat conduction problem with time fractional derivatives and generalization of thermos elasticity with some limiting cases using fractional calculus etc[11-21] has been studied widely in last decade from the research fraternities.

For the caputo fractional derivative [2]

$$D^{\alpha}f(t)=\begin{cases}\frac{1}{\Gamma n-\alpha}\int_0^t\frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, & \text{n-1} < \alpha < n;\\ \frac{df(t)}{dt}, & \text{n=1}.\end{cases}$$

with Laplace transform

$$L\{D^{\alpha}f(t);s\} = s^{\alpha}F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}f^{(k)}(0), \qquad n-1<\alpha<n$$

By considering this Fractional derivative and its Laplace operator the heat equaion

$$\Delta u(x,t) = \frac{\partial}{\partial t}u(x,t),$$

with the kernel as the basic solution for the heat equation

$$W(x,t) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{\|x\|^2}{4t}},\ (x,t)\in\mathbb{R}_{+}^{n+1}.$$

we consider the heat equation satisfying the deflection function

$$\nabla^2\nabla^2 w = -\frac{1}{(1-\nu)D}\nabla^2 M_T$$

here

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

and M_T as the thermal moment, v is the Poisson ratio, D is the flexural rigidity, the solution which was obtained by the Fourier, Hankel and Laplace transform successively is

$$w(r,t) = -\frac{1}{\sqrt{2\pi}} a_t E h \frac{1}{(1-\nu)D} \sum_{p=1}^{\infty} \sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p} \frac{1}{\beta_m} \times \frac{\pi}{\sqrt{2}} \frac{J_0(\beta_m b).Y_0(\beta_m b)}{\left[1 - \frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}} \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \frac{1}{k(\beta_m^2 + \eta_p^2)} [1 - E_\alpha(-k(\beta_m^2 + \eta_p^2)t^\alpha)].b_{pm}$$

here a_t and E are the coefficients of thermal expansion and Young modulus, η, β are the positive roots of the transcendental equations, J_0, Y_0, E_α are the Bessels and Mittag leffler functions, a,b ,r are the real parameters of the range in interval and the coefficient of the series is

$$b_{pm} = \Bigg[k.a.K_1(\beta_m, a) \int_{z'=0}^{h} K(\eta_p, z').f_1(z',t').dz' - k.b.K_1(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z').f_2(z',t').dz' + \sqrt{\frac{2}{\pi}} k.\eta_p. \int_{r'=a}^{b} r'.K_0(\beta_m, r').f_3(r',t').dr' + \sqrt{\frac{2}{\pi}} k.\eta_p.\cos(\eta_p h). \int_{r'=a}^{b} r'.K_0(\beta_m, r').f_4(r',t').dr' \Bigg]$$

In this article we will take the positive solution of heat equation which is non negative and with the help of some results we will find the convergence for such a solution.

Preliminaries

The Borel measure μ and total variation on $|\mu|$ with compact set K we will take finite and the Gauss Weirstrass integral [21] of measure as

$$W\mu(x,t) = \int_{\mathbb{R}^n} W(x-y,t)\,d\mu(y), \quad x \in \mathbb{R}^n,\ t \in (0,\infty),$$

which has known result[21]

$$D_{sym}\mu(x_0) := \lim_{r\to 0} \frac{\mu(B(x_0,r))}{m(B(x_0,r))} = L,$$

with

$$\lim_{t\to 0} W\mu(x_0,t) = L,$$

For the solution of the heat equation with the said boundary conditions we have one important result [12]

Theorem:[12] For the positive solution of heat equation

If

$$\beta'(x_0) = L$$

then

$$\lim_{\substack{(x,t)\to(x_0,0)\\(x,t)\in P(x_0,\alpha)}} u(x,t) = L.$$

and if

$$\lim_{t\to 0} u(x_0+\alpha_1\sqrt{t},t) = L = \lim_{t\to 0} u(x_0+\alpha_2\sqrt{t},t),$$

then

$$\beta'(x_0) = L$$

Lemma[19]: suppose μ is measure in the given real space such that $W\mu(x_0, y_0)$ is finite at some points then $W\mu$ is well define and the solution of heat equation.

and

$$\int_{\mathbb{R}^n} Wf(x,t)\, d\nu(x) = \int_{\mathbb{R}^n} W\nu(x,t) f(x)\, dx.$$

Definition: A function μ is said to have parabolic limit if

$$\lim_{\substack{(x,t)\to(x_0,0)\\ (x,t)\in \mathrm{P}(x_0,\alpha)}} u(x,t) = L,$$

where

$$\mathrm{P}(x_0,\alpha) = \{(x,t) \in \mathbb{R}^{n+1}_+ : \|x - x_0\|^2 < \alpha t\}.$$

Definition: A function μ has strong derivative if

$$\lim_{r\to 0} \frac{\mu(x_0 + rB)}{m(rB)} = L$$

for the open ball B in R

lemma[21]: A sequence of solution is said to converge a solution if

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \phi(y)\, d\mu_j(y) = \int_{\mathbb{R}^n} \phi(y)\, d\mu(y),$$

for all $\phi \in C_c(\mathbb{R}^n)$.

Main Result

With the help of preliminaries and previous lemmas we will prove the convergence of the positive solution at the boundary as follows

Theorems: let w is the positive solution of the heat equation

$$\nabla^2\nabla^2 w = -\frac{1}{(1-\nu)D}\nabla^2 M_T$$

having the form

$$w(r,t) = -\frac{1}{\sqrt{2\pi}} a_t E h \frac{1}{(1-\nu)D} \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} \frac{[\cos(\eta_p h)+1]}{\eta_p}\frac{1}{\beta_m} \times \frac{\pi}{\sqrt{2}} \frac{J_0(\beta_m b).Y_0(\beta_m b)}{\left[1-\frac{J_0^2(\beta_m b)}{J_0^2(\beta_m a)}\right]^{1/2}} \left[\frac{J_0(\beta_m r)}{J_0(\beta_m b)} - \frac{Y_0(\beta_m r)}{Y_0(\beta_m b)}\right] \frac{1}{k(\beta_m^2+\eta_p^2)}[1-E_\alpha(-k(\beta_m^2+\eta_p^2)t^\alpha)].b_{pm}$$

where

$$b_{pm} = \Bigg[k.a.K_1(\beta_m, a) \int_{z'=0}^{h} K(\eta_p, z').f_1(z', t').dz' \\ - k.b.K_1(\beta_m, b) \int_{z'=0}^{h} K(\eta_p, z').f_2(z', t').dz' \\ + \sqrt{\frac{2}{\pi}} k.\eta_p. \int_{r'=a}^{b} r'.K_0(\beta_m, r').f_3(r', t').dr' \\ + \sqrt{\frac{2}{\pi}} k.\eta_p. \cos(\eta_p h). \int_{r'=a}^{b} r'.K_0(\beta_m, r').f_4(r', t').dr' \Bigg]$$

consider w=u having the boundary measure μ is finite then following holds

If there exists $\eta > 0$, such that

$$\lim_{\substack{(x,t)\to(0,0) \\ (x,t)\in P(0,\eta)}} u(x,t) = L,$$

then the strong derivative of μ at zero is also equal to L.

Proof:

we consider the open ball the sequence of positive solutions converging to zero and quotient

$$M_j = \frac{\mu(r_j B_0)}{m(r_j B_0)}.$$

to prove $\{Mj\}$ is a bounded sequence and every convergent subsequence of $\{Mj\}$ converges to L

choose a positive real number s such that $B0$ is contained in $B(0, s)$. Then

$$M_j \leq \frac{\mu(r_j B(0,s))}{m(r_j B_0)} = \frac{\mu(r_j B(0,s))}{m(r_j B(0,s))} \times \frac{m(B(0,s))}{m(B_0)} \leq \frac{m(B(0,s))}{m(B_0)} M_{HL}(\mu)(0).$$

Since μ is the boundary measure for u

$$u(x,t) = W\mu(x,t)$$

by the consideration and information with us we have

$$\sup_{t<\beta} u(0,t^2) < \infty$$

and since the measure is finite It follows that $u(0,t^2)$ is a bounded function of $t \in (0,\infty)$.

Which Implies the boundedness.

to prove that every convergent subsequence of $\{Mj\}$ converges to L. We choose a convergent subsequence of $\{Mj\}$,

let

$$u_j(x,t) = u(r_j x, r_j^2 t),$$

$\{uj\}$ is a sequence of solutions of the heat equation, consider the map

$$(x,t) \mapsto \frac{\|x\|^2}{t},$$

Clearly, this map is continuous. As K is compact, image of K under this map is bounded and hence there exists a positive real number α such that

$$\frac{\|x\|^2}{t} < \alpha, \quad \text{for all } (x,t)$$

which follows

$$\sup_{(x,t)\in \mathrm{P}(0,\alpha)} u_j(x,t) \leq \sup_{(x,t)\in \mathrm{P}(0,\alpha)} u(x,t) \leq c_\alpha M_{HL}(\mu)(0).$$

Hence, $\{uj\}$ is locally bounded, we claim by this

$$v(x,t) = L = W(Lm)(x,t).$$

since the sequence converges to zero as k goes to infinity and u has a limit L

$$v(x_0,t_0) = \lim_{k\to\infty} u_{j_k}(x_0,t_0) = \lim_{k\to\infty} u(r_{j_k}x_0, r_{j_k}^2 t_0) = L,$$

with the help of lemma we can have the result now if

$$u_{j_k}(x,t) = u(r_{j_k}x, r_{j_k}^2 t) = (W\mu)(r_{j_k}x, r_{j_k}^2 t) = W(\mu_{r_{j_k}})(x,t).$$

This gives that $\{W(\mu_{r_{jk}})\}$ converges normally to $W(Lm)$ and It follows from Lemma that the sequence of measures $(\mu_{r_{jk}})$ converges to Lm in weak and hence by Lemma $(\mu_{r_{jk}}(B)$ converges to $Lm(B)$ for every ball . Therefore,

$$Lm(B_0) = \lim_{k\to\infty} \mu_{r_{j_k}}(B_0) = \lim_{k\to\infty} r_{j_k}{}^{-n}\mu(r_{j_k}B_0) = \lim_{k\to\infty} \frac{\mu(r_{j_k}B_0)}{m(r_{j_k}B_0)} m(B_0) = m(B_0)\lim_{k\to\infty} M_{j_k}.$$

This implies that the sequence $\{Mj_k\}$ converges to L and hence, so does $\{Mj\}$. This completes the proof.

Conclusion

For the said heat equation with fractional derivative, the measure of the Borel gives us the convergence of the positive solution of the heat equation on the boundary and its behavior convergence is observed, the brief discussion further will be elaborated in the continuous discussion for boundary value time fractional heat equation, the parabolic as well as strong derivative convergence and equality will be obtained further for the boundary value time fractional heat equation.

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